Mock exam sets October 2023

Always explain your answers. It is allowed to refer to definitions, lemmas and theorems from the lecture notes but not to other sources. All questions are independent and count equally so make sure you try each of them. Good luck!

0. How many elements does $\{\emptyset \setminus \emptyset\} \setminus \emptyset$ have?

1. Prove that if C and D are sets then $(C \cup D)^2 = C^2 \cup D^2 \cup (C \times D) \cup (D \times C)$.

2. Prove that if $a \in Y^X$ and $b \in Z^Y$ and $c \in W^Z$ are invertible functions then $(c \circ b \circ a)^{-1} = a^{-1} \circ b^{-1} \circ c^{-1}$.

3. Give an explicit example of an equivalence relation R on $\mathbb{Z}$ such that $\#(\mathbb{Z}/R) = 3$ and all equivalence classes have different cardinality. You should demonstrate why your R is indeed an equivalence relation.

4. **Explain why the following argument is invalid:** Consider two uncountable sets U, V a function $f : U \to V$ and a finite subset $X \subset U$ with the property that $\#X = 1000$ and $f(U) = f(X)$. Since
$$U = \{u \in U : f(u) \in f(X)\} = f^{-1}(f(X)) = X$$
we conclude that U and X have the same cardinality.

5. Prove by induction that for all natural numbers $n \geq 1$ we have
$$\bigcup_{k \in [n]} [k, k+1] \subseteq [0, \frac{n(n+1)}{2}]$$

6. Show that if Y, A, B are sets and $A \cap B = \emptyset$ then $Y^{A \cup B}$ has the same cardinality as $(Y^A) \times (Y^B)$.

7. We say an equivalence relation R on a set X is special if it satisfies the following property for all $x, y, z, w \in X$: If $x \sim y$ and $z \sim w$ then $x \sim w$. Prove that the only special equivalence relation on X is X^2.

8. Give an example of an invertible function that is the composition of two non-invertible functions. Prove that what you say is correct!

9. For a set X with more than one element define for any $p, q \in X$ the function $j_{p,q} \in X^X$ by $j_{p,q}(p) = q$, $j_{p,q}(q) = p$ and $j_{p,q}(x) = x$ if $x \notin \{p, q\}$. Also define $\mathcal{F} : X^{(X^X)} \to (X^X)^X$ by $(\mathcal{F}(u)(p))(q) = u(j_{p,q})$. Prove that $\mathcal{F}$ is not invertible.

Solutions

0. How many elements does $\{\emptyset \setminus \emptyset\} \setminus \emptyset$ have?
   One because for any set X we have $X \setminus \emptyset = X$ so $\{\emptyset \setminus \emptyset\} \setminus \emptyset = \{\emptyset\}$.

1. Prove that if C and D are sets then $(C \cup D)^2 = C^2 \cup D^2 \cup (C \times D) \cup (D \times C)$.
   By the double inclusion lemma it suffices to prove that $L \subseteq R$ and $L \supseteq R$ where $L = (C \cup D)^2$ and $R = C^2 \cup D^2 \cup (C \times D) \cup (D \times C)$. First to prove $L \subseteq R$ we take some $\ell \in L$ and note that $\ell = (x, y)$ with $x, y \in C \cup D$. There are four (not exclusive) possibilities: either $x, y \in C$ in which case $(x, y) \in C^2$ or $x \in C, y \in D$ in which case $(x, y) \in C \times D$ or $x \in D, y \in C$ in which case $(x, y) \in D \times C$ or finally $x, y \in D$ in which case $(x, y) \in D^2$. In all four cases we find that $\ell = (x, y) \in R$.

   Second, to prove $L \supseteq R$ we take some $r \in R$ and note that there are four possible options: either $r \in C^2$ or $r \in C \times D$ or $r \in D \times C$ or $r \in D^2$. In each of the four cases this means $r = (x, y)$ with x and y an element of at least one of the sets C and D. In other words $r = (x, y) \in (C \cup D)^2 = L$.

2. Prove that if $a \in Y^X$ and $b \in Z^Y$ and $c \in W^Z$ are invertible functions then $(c \circ b \circ a)^{-1} = a^{-1} \circ b^{-1} \circ c^{-1}$.
   Define $f = b \circ a$ and using Lemma 2.6 twice (and implicitly Lemma 2.3) from the text we have

   $$(c \circ b \circ a)^{-1} = (c \circ f)^{-1} = f^{-1} \circ c^{-1} = (b \circ a)^{-1} \circ c^{-1} = a^{-1} \circ b^{-1} \circ c^{-1}$$

   as required. The first equality is the definition of f, the second is Lemma 2.6, the third is the definition of f and the final equality follows again from 2.6.

3. Give an explicit example of an equivalence relation R on $\mathbb{Z}$ such that $\#(\mathbb{Z}/R) = 3$ and all equivalence classes have different cardinality. You should demonstrate why your R is indeed an equivalence relation.
   One option would be to set

   $$R = \{(-1, 1), (1, -1)\} \cup \{(a, b) \in \mathbb{Z} : |a| > 1, |b| > 1\} \cup \{(a, b) \in \mathbb{Z} : a = b\}$$

   This is indeed an equivalence relation because it satisfies the three properties: reflexivity is clear from the fact that $\{(a, b) \in \mathbb{Z} : a = b\} \subseteq R$, it shows that $a \sim a$ for all $a \in \mathbb{Z}$. Symmetry is also clear because if $(x, y) \in R$ then there are three options: $x = y$ or $|x| = |y| = 1$ or both $|x| > 1$ and $|y| > 1$. In the first case there is nothing to prove, in the second case we see explicitly that $(y, x) \in R$ and in the third case we also have $(y, x) \in R$ because of the middle set in the definition of R. A complete list of all the equivalence classes is $\mathbb{Z}/R = \{\bar{0}, \bar{1}, \bar{2}\}$. Indeed, no other integer is equivalent to 0 so $\#\bar{0} = 1$ and $\bar{1} = \{-1, 1\}$ has two elements. Finally the remaining equivalence class is not finite. (If it were finite then taking the union with the two finite sets $\bar{0}$ and $\bar{1}$ would still make it finite and we would have shown that $\mathbb{Z}$ itself is a finite set).

4. **Explain why the following argument is invalid:** Consider two uncountable sets U, V a function $f : U \to V$ and a finite subset $X \subset U$ with the property that $\#X = 1000$ and $f(U) = f(X)$. Since

$$U = \{u \in U : f(u) \in f(X)\} = f^{-1}(f(X)) = X$$

we conclude that U and X have the same cardinality.
It is not true that $f^{-1}(f(X)) = X$ when f is not injective. Since we assumed that $f(U) = f(X)$ and U is infinite while X is not, there exists some $x \in X$ and an element $u \in X^c$ such that $f(u) = f(x)$. This shows that f is not injective. More concretely we can see how the argument fails if we make some specific choice for U, V and f: take $U = V = \mathbb{N}^{\mathbb{N}}$ (which by Cantor's theorem is indeed uncountable). Pick some arbitrary subset $X \subset U$ that has cardinality 1000 and some special $p \in X$. Now define $f : U \to V$ by setting $f(u) = u$ if $u \in X$ and $f(u) = p$ if $u \notin X$. Then clearly $f(U) = f(X) = X$ and $f^{-1}(f(X)) = U$.

5. Prove by induction that for all natural numbers $n \geq 1$ we have

$$\bigcup_{k \in [n]} [k, k+1] \subseteq [0, \frac{n(n+1)}{2}] \tag{1}$$

Define S_m to be the statement that Equation (1) holds when $\mathbb{N} \ni m = n - 1$, Note that S_0 holds because in that case $n = 1$ so $k \in= [1] = \{0\}$ and the left hand side becomes $[0, 1]$ while the right hand side is $[0, 1]$ as well.
Now assume S_m is true for some $m \in \mathbb{N}$. We will demonstrate that then S_{m+1} is also true. Using $n = m + 1$ as before the left hand side is in the case of S_{m+1} equal to $\bigcup_{k \in [n+1]} [k, k+1] = [n+1, n+2] \cup \bigcup_{k \in [n]} [k, k+1] = [n+1, n+2] \cup [0, \frac{n(n+1)}{2}] \subseteq \frac{(n+1)(n+2)}{2}]$ because both $n+2$ and $\frac{n(n+1)}{2}$ are less than $\frac{(n+1)(n+2)}{2}$. In the final equality we used the induction hypothesis S_m. This finishes the proof.

6. Show that if Y, A, B are sets and $A \cap B = \emptyset$ then $Y^{A \cup B}$ has the same cardinality as $(Y^A) \times (Y^B)$.
Define a function $F : (Y^A) \times (Y^B) \to Y^{A \cup B}$ as follows. For $u : A \to Y$ and $v : B \to Y$ we define $F(u, v)$ to be the function on $A \cup B$ defined by $F(u, v)(x) = u(x)$ if $x \in A$ and $v(x)$ if $x \in B$. This F is well-defined because there are no elements in the domain that are both in A and in B. Moreover F is a bijection and hence invertible and that shows that $Y^{A \cup B}$ has the same cardinality as $(Y^A) \times (Y^B)$. To see that F is invertible we claim that F is both injective and surjective. To show injectivity consider elements (u, v) and (r, s) of $(Y^A) \times (Y^B)$ such that $F(u, v) = F(r, s)$. By definition of F this means that for all $a \in A$ we have $u(a) = r(a)$ and likewise $v(b) = s(b)$ for all $b \in B$. It follows that we must have $u = r$ and $v = s$ so $(u, v) = (r, s)$ as required. For surjectivity of F consider $g : A \cup B \to Y$ and denote by u and v the restrictions of g to A and B so

$u(a) = g(a)$ for all $a \in A$ and $v(b) = g(b)$ for all $b \in B$. Then $F(u, v) = g$ because for each $x \in A \cup B$ they give the same output. This concludes our proof because we showed that F is a bijection, hence invertible.

7. We say an equivalence relation R on a set X is special if it satisfies the following property for all $x, y, z, w \in X$: If $x \sim y$ and $z \sim w$ then $x \sim w$. Prove that the only special equivalence relation on X is X^2.
   If R is special and $(x, w) \in X^2$ then we claim $(x, w) \in R$ so that $R = X^2$. To this end choose $x = y$ and $z = w$. By reflexivity we have $x \sim x$ and $w \sim w$ so since R is special we conclude $x \sim w$, in other words $(x, w) \in R$.

8. Give an example of an invertible function that is the composition of two non-invertible functions. Prove that what you say is correct!
   Define $f : [1] \to [2]$ and $g : [2] \to [1]$ by $f(0) = 0$ and $g(0) = 0$ and $g(1) = 0$. Then $h = g \circ f = \mathrm{id}_{[1]}$ which is invertible while g is not injective (because $g(0) = g(1)$) and f is not surjective because 1 is not in the range of f.

9. For a set X with more than one element define for any $p, q \in X$ the function $j_{p,q} \in X^X$ by $j_{p,q}(p) = q, j_{p,q}(q) = p$ and $j_{p,q}(x) = x$ if $x \notin \{p, q\}$. Also define $\mathcal{F} : X^{(X^X)} \to (X^X)^X$ by $(\mathcal{F}(u)(p))(q) = u(j_{p,q})$. Prove that $\mathcal{F}$ is not invertible.
   We will show that $\mathcal{F}$ is not injective by producing two distinct functions $u, v \in X^{(X^X)}$ with the property $\sqcap = \sqsubseteq$. Notice that for any $p, q \in X$ the function $j_{p,q}$ is a bijection. Since X has at least two elements, say x_0 and x_1 we can define u by $u(f) = x_0$ when $f \in X^X$ is a bijection and $u(f) = x_1$ if f is not a bijection. Now define $v(f) = x_0$ for all $f \in X^X$. Then we have $u \neq v$ and yet $\mathcal{F}(u) = \mathcal{F}(v)$ because for any $p, q \in X$ we have $(\mathcal{F}(u)(p))(q) = u(j_{p,q}) = x_0 = v(j_{p,q}) = \mathcal{F}(v)$. This proves $\mathcal{F}$ is not injective and hence not invertible.